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Extended Birman–Wenzl algebra and Yang–Baxterization

Mo-Lin Ge† and Kang Xue‡§

† Theoretical Physics Section, Nankai Institute of Mathematics, Tianjin 300071, People's Republic of China
‡ Institute for Theoretical Physics, State University of New York at Stony Brook, Stony Brook, NY 11794-3840, USA

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Abstract. A new algebra $\mathscr{A}$ is presented by directly extending the Birman–Wenzl algebra. If a braid group representation with four distinct eigenvalues satisfies $\mathscr{A}$ then it can be Yang–Baxterized to derive the solution of the Yang–Baxter equation in terms of our standard scheme.

1. Introduction

Much progress has been made in deriving solutions of braid group representation (BGR) which may be considered as an appropriate limit of the Yang–Baxter equation (YBE) when the spectral parameter $x=\exp(-u)$ disappears. In this limit one obtains the YBE with no x dependence, also known as braid relation which solutions are BGRs. The standard BGR can be obtained through the representation theory of quantum algebra by many authors including Drinfeld [1], Jimbo [2–4], Reshetikhin [5] and by taking limit of statistical models as discussed in [6–8] or by direct calculation [9–11]. The non-standard BGRs can be computed in the similar way [12–18]. Some of them are super-extension of standard ones [17] whereas some are not [12–15], but they may preserve the quantum double [1, 19, 20].

In the previous works [21, 22] the following results have been made.

(1) The Yang–Baxterization prescription is established to incorporate the x dependence into the BGRs regardless whether or not BGRs are standard or non-standard, even regardless whether BGRs possess projectors or not.

(2) If BGRs obey certain algebra then the Yang–Baxterization works. This perception was first noticed by Jones [21]. For instance, for $N=2$ it is related to Hecke algebra, where N is the number of distinct eigenvalues of considered BGR. When $N=3$, as shown in [24] that the Birman–Wenzl algebra (BWA) sufficiently satisfies the Yang–Baxterization condition for the case (a)

$$f_3^+\theta_3^+ + f_3^-\theta_3^- + f_2\theta_2 + f_1^+\theta_1^+ + f_1^-\theta_1^- = 0 \tag{1.1}$$

where

$$\theta_3^\pm = S_1^{\pm1}S_2^{\mp1}S_1^{\pm1} - S_2^{\pm1}S_1^{\mp1}S_2^{\pm1}$$

$$\theta_2 = S_1S_2^{-1} - S_2S_1^{-1} + S_2^{-1}S_1 - S_1^{-1}S_2$$

$$\theta_1^\pm = S_1^{\pm1} - S_2^{\pm1}$$

§ Permanent address: Department of Physics, Northeast Normal University, Changchun 130024, People's Republic of China.

and $f_3^\pm, f_2, f_1^\pm$ are given by

$$f_3^+ = \lambda_3^{-1} \qquad f_3^- = -\lambda_1$$
$$f_2 = -\lambda_2^{-1}\lambda_3^{-1}(\lambda_1+\lambda_2)(\lambda_2+\lambda_3)$$
$$f_1^\pm = \mp\lambda_2^{\mp1} f_2.$$

So that, in general, the Yang–Baxterization and its relationship with BWA have been solved for $N=3$.

(3) However, for $N=4$ there has not yet been a satisfactory algebra in accordance with the Yang–Baxterization for $N=4$. Wadati *et al* [23] made progress in giving a spin model, but it does not connect with the Yang–Baxterization. Hinted by all the known results the Yang–Baxterization should be related to certain algebra.

In this paper we shall, in parallel to BWA, establish a new algebra $\mathscr{A}$ induced by Yang–Baxterization for $N=4$, which is an extension but more complicated.

2. Main results for algebra $\mathscr{A}(E, T, T, T^{-1}, I)$

We want to derive trigonometric solutions of YBE

$$\check{R}_{12}(x)\check{R}_{23}(xy)\check{R}_{12}(y) = \check{R}_{23}(y)\check{R}_{12}(xy)\check{R}_{23}(x) \tag{2.1}$$

through the Yang–Baxterization for given BGR T satisfying

$$T_{12}T_{23}T_{12} = T_{23}T_{12}T_{23} \tag{2.2}$$

with

$$\check{R}(x=0) = \text{constant}\times T \qquad \check{R}(x=1) = \text{constant}\times I \qquad \check{R}(x)\check{R}(x^{-1}) = \rho(x)I. \tag{2.3}$$

As is known that BGR can be expressed by the form

$$T = \sum_{\nu=1}^{4} \lambda_\nu P_\nu \tag{2.4}$$

for $N=4$, where P_ν are the projectors and λ_ν are distinct eigenvalues of T, and the suggested form of $\check{R}(x)$ is given by [22]

$$\check{R}(x) = A(x)T^2 + B(x)T + C(x)I + D(x)T^{-1} \tag{2.5}$$

where

$$A(x) = (\lambda_2\lambda_3\lambda_4)^{-1}(\lambda_4-\lambda_1)^{-1}(\lambda_2\lambda_4-\lambda_1\lambda_3)x(x-1)$$

$$B(x) = -\lambda_4^{-1}(x-1)\left[1+\frac{\lambda_2\lambda_3}{\lambda_4-\lambda_1}((\lambda_2+\lambda_3)(\lambda_2\lambda_4-\lambda_1\lambda_3)-\lambda_2\lambda_4^2-\lambda_1^2\lambda_3)\right]$$

$$C(x) = \frac{(\lambda_2\lambda_3\lambda_4)^{-1}}{\lambda_4-\lambda_1}\{[(\lambda_1+\lambda_2)(\lambda_3+\lambda_4)(\lambda_2\lambda_4-\lambda_1\lambda_3)+\lambda_2\lambda_3(\lambda_4^2-\lambda_1^2)$$
$$+\lambda_1\lambda_4(\lambda_3\lambda_4-\lambda_1\lambda_2)]x^2+[\lambda_3^2\lambda_4(\lambda_1+\lambda_2)-\lambda_1\lambda_2^2(\lambda_3+\lambda_4)]x\}$$

$$D(x) = \lambda_1(x+(\lambda_4-\lambda_1)^{-1}(\lambda_3-\lambda_2))x(x-1)$$

and

$$T^3 = \left(\sum_{\nu=1}^{4}\lambda_\nu\right) - \left(\sum_{\mu<\nu}^{4}\lambda_\mu\lambda_\nu\right)T + \left(\sum_{\tau<\mu<\nu}^{4}\lambda_\tau\lambda_\mu\lambda_\nu\right)I - \left(\prod_{\nu=1}^{4}\lambda_\nu\right)T^{-1} \tag{2.7}$$

where I is the unity in $V\times V$. We would like to emphasize that there is not a general theory of state what kind of BGRs make equation (2.5) work. Our task is to find an algebra $\mathscr{A}$ such that equation (2.5) sufficiently satisfied if BGR obeys $\mathscr{A}$.

Let us define

$$T_i = I_1\otimes\ldots\otimes I_{i-1}\otimes T\otimes I_{i+2}\otimes\ldots\otimes I_n \tag{2.8}$$

$$E_i = I_1\otimes\ldots\otimes I_{i-1}\otimes E\otimes I_{i+2}\otimes\ldots\otimes I_n \tag{2.9}$$

where

$$E=\alpha P_4. \tag{2.10}$$

It is worth noting that equation (2.10) is a direct extension of BWA where $S-S^{-1}\sim P_3$ for S being BGR and P_3 is only related to the largest submatrix in a block diagonal structure of BGR [24].

Consider that the P_4 can be represented by

$$P_4=\prod_{\nu=1}^{3}\frac{(T-\lambda_\nu)}{(\lambda_4-\lambda_\nu)} \tag{2.11}$$

we thus rewrite the reduction relation (2.7) in the form

$$T_i^2=\beta_1 T_i-\beta_2 I_i+\beta_3 T_i^{-1}+\lambda_4^{-1}\left(\prod_{\nu=1}^{3}(\lambda_4-\lambda_\nu)\right)\alpha^{-1}E_i \tag{2.12}$$

with

$$\beta_1=\sum_{\nu=1}^{3}\lambda_\nu \qquad \beta_2=\sum_{\mu<\nu}^{3}\lambda_\mu\lambda_\nu \qquad \beta_3=\prod_{\nu=1}^{3}\lambda_\nu .$$

To search for algebra $\mathscr{A}(E,T,T^{-1},I)$ we recast equation (2.5) to the form

$$\check{R}(x)=\lambda_1 x^3 T^{-1}+\sum_{j=1}^{2}x^j M^{(j)}+\lambda_4^{-1}T \tag{2.13}$$

where

$$M^j=a_jT^{-1}+b_jI+c_jT+d_jE \quad (j=1,2) \tag{2.14}$$

in which

$$\begin{aligned}
&a_1=-\lambda_1\lambda_3\lambda_4^{-1} \qquad b_1=\lambda_3\lambda_4^{-1}b_2 \qquad c_1=-\lambda_1^{-1}\lambda_3^{-1}a_2\\
&a_2=\lambda_1\lambda_4^{-1}(\lambda_3-\lambda_4) \qquad b_2=\lambda_2^{-1}\lambda_3^{-1}(\lambda_1+\lambda_2)(\lambda_2+\lambda_3) \qquad c_2=-\lambda_3^{-1}\\
&d_2=-d_1=\lambda_4^{-2}\beta_3^{-1}(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)(\lambda_2\lambda_4-\lambda_1\lambda_3)\alpha^{-1}.
\end{aligned} \tag{2.15}$$

Our main result is the following theorem.

Theorem. If (i) BGR T satisfies equation (2.7) and CP-invariance

$$T_{cd}^{ab}=T_{-d-c}^{-b-a} \tag{2.16}$$

(ii) E is defined by equation (2.10), and (iii) the eigenvalue λ_4 appears only in the largest sub-block matrix once, then there exists an algebra $\mathscr{A}(E,T,T^{-1},I)$ associated with equation (2.5). Besides equation (2.12) the algebra $\mathscr{A}$ contains the following relations

$$T_iT_{i\pm1}T_i=T_{i\pm1}T_iT_{i\pm1} \tag{2.17}$$

$$T_iT_j=T_jT_i \qquad E_iE_j=E_jE_i \quad (|i-j|>1) \tag{2.18}$$

$$E_i^2=\alpha E_i \qquad E_iT_i=T_iE_i=\lambda_4E_i \tag{2.19}$$

$$E_iE_{i\pm1}E_i=E_i \qquad E_iT_{i\pm1}E_i=\mu\lambda_4^{-1}E_i \tag{2.20}$$

and the descendent of equations (2.12), (2.17)-(2.20)

$$E_iT_{i\pm1}^{\pm1}T_i^{\pm1}=T_{i\pm1}^{\pm1}T_i^{\pm1}E_{i\pm1}=\mu^{\pm1}E_iE_{i\pm1} \tag{2.21}$$

$$E_iE_{i\pm1}T_i^{\pm}=\mu^{\pm1}E_iT_{i\pm1}^{\mp1} \qquad T_i^{\pm1}E_{i\pm1}E_i=\mu^{\pm1}T_{i\pm1}^{\mp1}E_i \tag{2.22}$$

$$T_iE_{i\pm1}T_i=\mu^2T_{i\pm1}^{-1}E_iT_{i\pm1}^{-1} \qquad E_iT_{i\pm1}^{-1}E_i=\mu^{-1}\lambda_4E_i \tag{2.23}$$

$$E_iT_{i\pm1}T_i^{-1}=\beta_3^{-1}\left[\mu^2E_iT_{i\pm1}^{-1}-\mu\beta_1E_iE_{i\pm1}+\beta_2E_iT_{i\pm1}-\mu\lambda_4^{-2}\prod_{\nu=1}^{3}(\lambda_4-\lambda_\nu)\alpha^{-1}E_i\right] \tag{2.24}$$

$$T_i^{-1}T_{i\pm1}E_i=\beta_3^{-1}\left[\mu^2T_{i\pm1}^{-1}E_i-\mu\beta_1E_{i\pm1}E_i+\beta_2T_{i\pm1}E_i-\mu\lambda_4^{-2}\prod_{\nu=1}^{3}(\lambda_4-\lambda_\nu)\alpha^{-1}E_i\right] \tag{2.25}$$

$$E_iT_{i\pm1}^{-1}T_i=\beta_1E_iT_{i\pm1}^{-1}-\mu^{-1}\beta_2E_iE_{i\pm1}+\mu^{-2}\beta_3T_{i\pm1}E_i+\mu^{-1}\lambda_4^{-1}\prod_{\nu=1}^{3}(\lambda_4-\lambda_\nu)\alpha^{-1}E_i \tag{2.26}$$

$$T_iT_{i\pm1}^{-1}E_i=\beta_1T_{i\pm1}^{-1}E_i-\mu^{-1}\beta_2E_{i\pm1}E_i+\mu^{-2}\beta_3T_{i\pm1}E_i+\mu^{-1}\lambda_4^{-1}\prod_{\nu=1}^{3}(\lambda_4-\lambda_\nu)\alpha^{-1}E_i \tag{2.27}$$

$$\sum_{\varepsilon=\pm}\sum_{k=1,3,4}h_k^\varepsilon\Theta_k^\varepsilon+h_2\Theta_2+h_5\Theta_5+h_6\Theta_6=0 \tag{2.28}$$

where

$$\begin{aligned}
&\Theta_1^{+}=T_i^{\pm1}T_{i\pm1}^{\mp1}T_i^{\pm1}-T_{i\pm1}^{\pm1}T_i^{\mp1}T_{i\pm1}^{\pm1}\\
&\Theta_2=T_i^{-1}T_{i\pm1}+T_{i\pm1}T_i^{-1}-T_{i\pm1}^{-1}T_i-T_iT_{i\pm1}^{-1}\\
&\Theta_3^{\pm}=T_i^{\pm1}-T_{i\pm1}^{\pm1}\\
&\Theta_4^{\pm}=E_iT_{i\pm1}^{\pm1}+T_{i\pm1}^{\pm1}E_i-E_{i\pm1}T_i^{\pm1}-T_i^{\pm1}E_{i\pm1}\\
&\Theta_5=T_iE_{i\pm1}T_i-T_{i\pm1}E_iT_{i\pm1}\\
&\Theta_6=E_i-E_{i\pm1}\\
&h_1^{+}=\lambda_3^{-1} \qquad h_1^{-}=-\lambda_1 \qquad h_2=b_2 \qquad h_3^{\pm}=\pm\lambda_2^{\mp1}b_2\\
&h_4^{+}=\lambda_1^{-1}\lambda_2\lambda_3^{-2}\lambda_4d_2 \qquad h_4^{-}=-\lambda_3^{-1}\lambda_4\beta_3^{-1}\mu^2d_2 \qquad h_5=\lambda_1^{-1}\lambda_3^{-2}\lambda_4d_2\\
&h_6=\lambda_3^{-1}\lambda_4^{-1}\left(\prod_{\nu=1}^{3}(\lambda_4-\lambda_\nu)\right)\alpha^{-1}(2\mu\beta_3^{-1}d_2-\lambda_1^{-1}b_2)-\lambda_1^{-1}\lambda_3^{-2}\lambda_4\mu d_2^2
\end{aligned} \tag{2.29}$$

The details of proof will be given in the next section. Let us first make a brief interpretation to equations (2.17)-(2.29). The equation (2.10) means that a considered BGR T possesses block diagonal structure

$$T=\text{diag block}(T_m, T_{m-1}, \ldots, T_1, T_0, T_1', \ldots, T_{m-1}', T_m') \tag{2.30}$$

where T_0 stands for the sub-matrix with the largest size. If the eigenvalue λ_4 appears in T_0 only then the P_4 is related to T_0. The situation is similar to BWA where the eigenvalue λ_3 appears in its T_0. We thus define E proportional to P_4 and equations (2.19)-(2.23) can be followed. Only the parameters α and μ are left to be determined. The equations (2.24)-(2.27) are the consequence of equation (2.12) combining with equations (2.17)-(2.23). All these relations are a direct extension of BWA. However equations (2.17)-(2.27) are not closed. One needs a relation concerning $T_iT_{i\pm1}^{-1}T_i$ and $T_i^{-1}T_{i\pm1}T_i^{-1}$ to form algebra $\mathscr{A}$. The equation (2.28) is nothing but the wanted one. Substituting equations (2.13) and (2.14) into equation (2.1) after lengthy calculations we derive equations (2.28) and (2.29) that together with equations (2.17)-(2.27) complete all the relations forming algebra $\mathscr{A}(E, T, T^{-1}, I)$. It is worth noting that the situation here is different from BWA where the relation equation (1.1) corresponding to equation (2.28) is automatically satisfied by BWA [24]. In the next section we shall give the proof of the above results.

3. Proof of the main results

Consider that E is only related to T_0, we can write E in the following form

$$E=\sum_{a,b} r_a r_{-b} e_{a-b}\otimes e_{-ab} \tag{3.1}$$

where a, $b\in[-(N-1)/2, -(N-1)/2+1,\ldots,(N-1)/2]$, and $(e_{ab})_{ik}=\delta_{ai}\delta_{bk}$. If the coefficients r_a satisfy relations:

$$r_a r_{-a}=1 \qquad \sum_a r_{-a}^2 T_{ab}^{ab}=\mu\lambda_4^{-1} \tag{3.2}$$

then equations (2.20) can be obtained. In terms of equations (2.20) and (2.12) it is not difficult to derive equations (2.21)-(2.27). The difficulty is in determining the parameters α and μ and deriving equation (2.28) that are made by substituting (2.13) into the YBE (2.1) and employing equation (2.17). We then derive the following set of equations

$$M_{12}^{(2)}T_{23}^{-1}M_{12}^{(2)}+\lambda_1T_{12}^{-1}M_{23}^{(2)}T_{12}^{-1}=M_{23}^{(2)}T_{12}^{-1}M_{23}^{(2)}+\lambda_1T_{23}^{-1}M_{12}^{(2)}T_{23}^{-1} \tag{3.3}$$

$$T_{12}^{-1}M_{23}^{(2)}M_{12}^{(2)}+M_{12}^{(2)}T_{23}^{-1}M_{12}^{(1)}=M_{23}^{(2)}M_{12}^{(2)}T_{23}^{-1}+M_{23}^{(1)}T_{12}^{-1}M_{23}^{(2)} \tag{3.4a}$$

$$M_{12}^{(2)}M_{23}^{(2)}T_{12}^{-1}+M_{12}^{(1)}T_{23}^{-1}M_{12}^{(2)}=T_{23}^{-1}M_{12}^{(2)}M_{23}^{(2)}+M_{23}^{(2)}T_{12}^{-1}M_{23}^{(1)} \tag{3.4b}$$

$$T_{12}^{-1}M_{23}^{(2)}M_{12}^{(1)}+\lambda_4^{-1}M_{12}^{(2)}T_{23}^{-1}T_{12}=M_{23}^{(1)}M_{12}^{(2)}T_{23}^{-1}+\lambda_4^{-1}T_{23}T_{12}^{-1}M_{23}^{(2)} \tag{3.5a}$$

$$M_{12}^{(1)}M_{23}^{(2)}T_{12}^{-1}+\lambda_4^{-1}T_{12}T_{23}^{-1}M_{12}^{(2)}=T_{23}^{-1}M_{12}^{(2)}M_{23}^{(1)}+\lambda_4^{-1}M_{23}^{(2)}T_{12}^{-1}T_{23} \tag{3.5b}$$

$$\begin{aligned}&M_{12}^{(2)}M_{23}^{(2)}M_{12}^{(2)}+\lambda_1M_{12}^{(1)}T_{23}^{-1}M_{12}^{(1)}+\lambda_1^2T_{12}^{-1}M_{23}^{(1)}T_{12}^{-1}\\&\qquad=M_{23}^{(2)}M_{12}^{(2)}M_{12}^{(2)}+\lambda_1M_{23}^{(1)}T_{12}^{-1}M_{23}^{(1)}+\lambda_1^2T_{23}^{-1}M_{12}^{(1)}T_{23}^{-1}\end{aligned} \tag{3.6}$$

$$\begin{aligned}&M_{12}^{(2)}M_{23}^{(2)}M_{12}^{(1)}+\lambda_1T_{12}^{-1}M_{23}^{(1)}M_{12}^{(2)}+\lambda_1\lambda_4^{-1}M_{12}^{(1)}T_{23}^{-1}T_{12}\\&\qquad=M_{23}^{(1)}M_{12}^{(2)}M_{23}^{(2)}+\lambda_1M_{23}^{(2)}M_{12}^{(1)}T_{23}^{-1}+\lambda_1\lambda_4^{-1}T_{23}T_{12}^{-1}M_{23}^{(1)}\end{aligned} \tag{3.7a}$$

$$\begin{aligned}&M_{12}^{(1)}M_{23}^{(2)}M_{12}^{(2)}+\lambda_1M_{12}^{(2)}M_{23}^{(1)}T_{12}^{-1}+\lambda_1\lambda_4^{-1}T_{12}T_{23}^{-1}M_{12}^{(1)}\\&\qquad=M_{23}^{(2)}M_{12}^{(2)}M_{23}^{(1)}+\lambda_1T_{23}^{-1}M_{12}^{(1)}M_{23}^{(2)}+\lambda_1\lambda_4^{-1}M_{23}^{(1)}T_{12}^{-1}T_{23}\end{aligned} \tag{3.7b}$$

$$\lambda_1 T_{12}^{-1} M_{23}^{(1)} M_{12}^{(1)} + \lambda_4^{-1} M_{12}^{(2)} M_{23}^{(2)} T_{12} = \lambda_1 M_{23}^{(1)} M_{12}^{(1)} T_{23}^{-1} + \lambda_4^{-1} T_{23} M_{12}^{(2)} M_{23}^{(2)} \tag{3.8a}$$

$$\lambda_1 M_{12}^{(1)} M_{23}^{(1)} T_{12}^{-1} + \lambda_4^{-1} T_{12} M_{23}^{(2)} M_{12}^{(2)} = \lambda_1 T_{23}^{-1} M_{12}^{(1)} M_{23}^{(1)} + \lambda_4^{-1} M_{23}^{(2)} M_{12}^{(2)} T_{23} \tag{3.8b}$$

$$\begin{aligned} M_{12}^{(2)} M_{23}^{(1)} M_{12}^{(2)} + M_{12}^{(1)} M_{23}^{(2)} M_{12}^{(1)} + \lambda_1^2 \lambda_4^{-1} T_{12}^{-1} T_{23} T_{12}^{-1} + \lambda_1 \lambda_4^{-2} T_{12} T_{23}^{-1} T_{12} \\ = M_{23}^{(2)} M_{12}^{(1)} M_{12}^{(2)} + M_{23}^{(1)} M_{12}^{(2)} M_{23}^{(1)} + \lambda_1^2 \lambda_4^{-1} T_{23}^{-1} T_{12} T_{23}^{-1} \\ + \lambda_1 \lambda_4^{-2} T_{23} T_{12}^{-1} T_{23} \end{aligned} \tag{3.9}$$

$$\begin{aligned} M_{12}^{(2)} M_{23}^{(1)} M_{12}^{(1)} + \lambda_4^{-1} M_{12}^{(1)} M_{23}^{(2)} T_{12} + \lambda_1 \lambda_4^{-1} T_{12}^{-1} T_{23} M_{12}^{(2)} \\ = M_{23}^{(1)} M_{12}^{(1)} M_{23}^{(2)} + \lambda_4^{-1} T_{23} M_{12}^{(2)} M_{23}^{(1)} + \lambda_1 \lambda_4^{-1} M_{23}^{(2)} T_{12} T_{23}^{-1} \end{aligned} \tag{3.10a}$$

$$\begin{aligned} M_{12}^{(1)} M_{23}^{(1)} M_{12}^{(2)} + \lambda_4^{-1} T_{12} M_{23}^{(2)} M_{12}^{(1)} + \lambda_1 \lambda_4^{-1} M_{12}^{(2)} T_{23} T_{12}^{-1} \\ = M_{23}^{(2)} M_{12}^{(1)} M_{23}^{(1)} + \lambda_4^{-1} M_{23}^{(1)} M_{12}^{(2)} T_{23} + \lambda_1 \lambda_4^{-1} T_{23}^{-1} T_{12} M_{23}^{(2)} \end{aligned} \tag{3.10b}$$

$$\begin{aligned} M_{12}^{(1)} M_{23}^{(1)} M_{12}^{(1)} + \lambda_4^{-1} M_{12}^{(2)} T_{23} M_{12}^{(2)} + \lambda_4^{-2} T_{12} M_{23}^{(2)} T_{12} \\ = M_{23}^{(1)} M_{12}^{(1)} M_{23}^{(1)} + \lambda_4^{-1} M_{23}^{(2)} T_{12} M_{23}^{(2)} + \lambda_4^{-2} T_{23} M_{12}^{(2)} T_{23} \end{aligned} \tag{3.11}$$

$$M_{12}^{(2)} M_{23}^{(1)} T_{12} + \lambda_1 T_{12}^{-1} T_{23} M_{12}^{(1)} = T_{23} M_{12}^{(1)} M_{23}^{(2)} + \lambda_1 M_{23}^{(1)} T_{12} T_{23}^{-1} \tag{3.12a}$$

$$T_{12} M_{23}^{(1)} M_{12}^{(2)} + \lambda_1 M_{12}^{(1)} T_{23} T_{12}^{-1} = M_{23}^{(2)} M_{12}^{(1)} T_{23} + \lambda_1 T_{23}^{-1} T_{12} M_{23}^{(1)} \tag{3.12b}$$

$$M_{12}^{(1)} M_{23}^{(1)} T_{12} + M_{12}^{(2)} T_{23} M_{12}^{(1)} = T_{23} M_{12}^{(1)} M_{23}^{(1)} + M_{23}^{(1)} T_{12} M_{23}^{(2)} \tag{3.13a}$$

$$T_{12} M_{23}^{(1)} M_{12}^{(1)} + M_{12}^{(1)} T_{23} M_{12}^{(2)} = M_{23}^{(1)} M_{23}^{(1)} T_{23} + M_{23}^{(2)} T_{12} M_{23}^{(1)} \tag{3.13b}$$

$$M_{12}^{(1)} T_{23} M_{12}^{(1)} + \lambda_4^{-1} T_{12} M_{23}^{(1)} T_{12} = M_{23}^{(1)} T_{12} M_{23}^{(1)} + \lambda_4^{-1} T_{23} M_{12}^{(1)} T_{23} \,. \tag{3.14}$$

Since T satisfies CP-invariance it holds

$$(A_{12} B_{23} C_{12})_{def}^{abc} = (A_{23} B_{12} C_{23})_{-f-e-d}^{-c-b-d} \tag{3.15}$$

for any BGRs. It follows that any two equations appearing in the pairs given by equations (3.4), (3.5), (3.7), (3.8), (3.10), (3.12), (3.13) are equivalent to each other. Using the relation

$$M^{(1)} + M^{(2)} = -\lambda_1 T^{-1} + \left(1 + \frac{\lambda_1}{\lambda_2}\right)\left(1 + \frac{\lambda_2}{\lambda_3}\right)\left(1 + \frac{\lambda_3}{\lambda_4}\right) I - \lambda_4^{-1} T \tag{3.16}$$

it follows

$$\begin{aligned} &\text{Eq (3.3)} + \text{Eq (3.4)} + \text{Eq (3.5)} = 0 \\ &\text{Eq (3.12)} + \text{Eq (3.13)} + \text{Eq (3.14)} = 0 \\ &\lambda_1 \text{ Eq (3.5)} + \text{Eq (3.8)} + \lambda_4^{-1} \text{ Eq (3.12)} = 0 \\ &\text{Eq (3.6)} + \text{Eq (3.8)} = \text{Eq (3.10)} + \text{Eq } \lambda_4^{-1}\text{(3.13)} \\ &\lambda_1 \text{ Eq (3.4)} + \text{Eq (3.7)} + \text{Eq (3.10)} + \lambda_4^{-1} \text{ Eq (3.13)} = 0 \\ &\text{Eq (3.9)} = \lambda_1 \text{ Eq (3.4)} + \text{Eq (3.8)} + \lambda_4^{-1} \text{ Eq (3.13)} \\ &\text{Eq (3.8)} + \text{Eq (3.10)} + \text{Eq (3.11)} + \lambda_4^{-1} \text{ Eq (3.13)} = 0 \end{aligned} \tag{3.17}$$

namely, only five equations in (3.4)-(3.14) are independent. We choose the following equations as the independent ones.

$$M_{12}^{(2)}T_{23}^{-1}M_{12}^{(2)}+\lambda_1T_{12}^{-1}M_{23}^{(2)}T_{12}^{-1}=M_{23}^{(2)}T_{12}^{-1}M_{23}^{(2)}+\lambda_1T_{23}^{-1}M_{12}^{(2)}T_{23}^{-1} \tag{3.18}$$

$$T_{12}^{-1}M_{23}^{(2)}M_{12}^{(2)}+M_{12}^{(2)}T_{23}^{-1}M_{12}^{(1)}=M_{23}^{(2)}M_{12}^{(2)}T_{23}^{-1}+M_{23}^{(1)}T_{12}^{-1}M_{23}^{(2)} \tag{3.19}$$

$$\lambda_1T_{12}^{-1}M_{23}^{(1)}M_{12}^{(1)}+\lambda_4^{-1}M_{12}^{(2)}M_{23}^{(2)}T_{12}=\lambda_1M_{23}^{(1)}M_{12}^{(1)}T_{23}^{-1}+\lambda_4^{-1}T_{23}M_{12}^{(2)}M_{23}^{(2)} \tag{3.20}$$

$$\begin{aligned}M_{12}^{(2)}M_{23}^{(1)}M_{12}^{(1)}&+\lambda_4^{-1}M_{12}^{(1)}M_{23}^{(2)}T_{12}+\lambda_1\lambda_4^{-1}T_{12}^{-1}T_{23}M_{12}^{(2)}\\&=M_{23}^{(1)}M_{12}^{(1)}M_{23}^{(2)}+\lambda_4^{-1}T_{23}M_{12}^{(2)}M_{23}^{(1)}+\lambda_1\lambda_4^{-1}M_{23}^{(2)}T_{12}T_{23}^{-1}\end{aligned} \tag{3.21}$$

$$M_{12}^{(1)}T_{23}M_{12}^{(1)}+\lambda_4^{-1}T_{12}M_{23}^{(1)}T_{12}=M_{23}^{(1)}T_{12}M_{23}^{(1)}+\lambda_4^{-1}T_{23}M_{12}^{(1)}T_{23}. \tag{3.22}$$

If BGR T satisfies equations (3.18)-(3.22) then the YBE (2.1) is satisfied.

The equations (3.18)-(3.22) provide the over constraints to the algebra since relations equations (2.17)-(2.27) have covered all the closed property of $\mathscr{A}(E, T, T^{-1}, I)$ except only one more relation concerning $T_iT_{i\pm1}^{-1}T_i$ and $T_i^{-1}T_{i\pm1}T_i^{-1}$. We thus have to choose the parameters α and μ such that all the five equations (3.18)-(3.22) are equivalent, namely only one relation is independent and is just equation (2.28) with equation (2.29). In the following we shall verify this point.

Substituting equation (2.14) into equation (3.18) and (3.22) we obtain

$$\begin{aligned}c_2^2\Theta_1^++\lambda_1c_2\Theta_1^--b_2c_2\Theta_2-b_2^2\Theta_3^-+\lambda_1b_2(T_i^{-2}-T_{i\pm1}^{-2})&\\+c_2d_2(E_iT_{i\pm1}^{-1}T_i+T_iT_{i\pm1}^{-1}E_i-E_{i\pm1}T_i^{-1}T_{i\pm1}-T_{i\pm1}T_i^{-1}E_{i\pm1})&\\+b_2d_2\Theta_4^--\lambda_1\mu^{-2}\Theta_5+\lambda_4\mu^{-1}d_2^2\Theta_6=0&\end{aligned} \tag{3.23}$$

and

$$\begin{aligned}\lambda_4^{-1}a_1\Theta_1^2+a_1^2\Theta_1^-+a_1b_1\Theta_2-b_1^2\Theta_3^++\lambda_4^{-1}b_1(T_i^2-T_{i\pm1}^2)&\\+a_1d_1(E_iT_{i\pm1}T_i^{-1}+T_i^{-1}T_{i\pm1}E_i-E_{i\pm1}T_iT_{i\pm1}^{-1}-T_{i\pm1}^{-1}T_iE_{i\pm1})&\\+b_1d_1\Theta_4^++\lambda_4^{-1}d_1\Theta_5+\lambda_4^{-1}\mu\Theta_6=0.&\end{aligned} \tag{3.24}$$

After calculations we find that when

$$\begin{aligned}&\mu^2=-\lambda_1^2\lambda_3^3\lambda_4^{-1}\\&\alpha=\lambda_1\lambda_2^{-1}\lambda_4^{-2}\mu^{-1}(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)(\lambda_2\lambda_4-\lambda_1\lambda_3)b_2^{-1}\end{aligned} \tag{3.25}$$

then equations (3.23) and (3.24) are equivalent to each other. Next by substituting equation (2.14) into equations (3.19), (3.20) and using equation (3.25) one gets

$$\lambda_1\lambda_3\lambda_4^{-1}\ \mathrm{Eq}\,(3.19)+\mathrm{Eq}\,(3.20)=0 \tag{3.26}$$

i.e. equations (3.19) and (3.20) are equivalent to each other. Similarly, one can verify that

$$\begin{aligned}&\lambda_1\lambda_3^2\lambda_4^{-2}\ \mathrm{Eq}\,(3.19)+\mathrm{Eq}\,(3.21)=0\\&c_1\ \mathrm{Eq}\,(3.18)-c_2\ \mathrm{Eq}\,(3.19)=0.\end{aligned} \tag{3.27}$$

In summary we conclude that there is only one independent equation among equations (3.18)-(3.22) if the parameters α and μ are given by equation (3.25). Choosing equation (3.22) as the independent relation we can simplify it in following form

$$\lambda_4^{-1}a_1\Theta_1^+ + a_1^2\Theta_1^- + a_1b_1\Theta_2 + (\lambda_4^{-1}\beta_1 - b_1)b_1\Theta_3^+ + \lambda_4^{-1}\beta_3b_1\Theta_3^-$$
$$+(\beta_3^{-1}\beta_2a_1 + b_1)d_1\theta_4^+ + \beta_3^{-1}\mu^2a_1b_1\Theta_4^- + \lambda_4^{-1}d_1\Theta_5$$
$$+\left\{\lambda_4^{-2}\left(\prod_{\nu=1}^{3}(\lambda_4-\lambda_\nu)\right)\alpha^{-1}(b_1 - 2\mu\beta_3^{-1}a_1d_1) + \lambda_4^{-1}\mu d_1^2\right\}\Theta_6 = 0 \qquad (3.28)$$

that can be recast into equation (2.28). We thus complete the proof.

4. Conclusion

We have presented an algebra $\mathscr{A}(E, T, T^{-1}, I)$ forming by equations (2.17)-(2.29) and verified that if BGR T satisfies $\mathscr{A}$ then $\check{R}(x)$ given by equation (2.13) solves YBE. Similar to BWA which is closely related to the Yang-Baxterization of T with three distinct eigenvalues the algebra $\mathscr{A}$ is induced by the same problem but with four distinct eigenvalues.

The simplest example satisfying $\mathscr{A}$ is BGR associated with 4-dimensional representation (spin $\frac{3}{2}$) of $SU_q(2)$. In the case we have

$$\lambda_1 = 1 \qquad \lambda_2 = -t^3 \qquad \lambda_3 = t^5 \qquad \lambda_4 = -t^6.$$

By taking $\mu = -(-\lambda_1^2\lambda_3^3\lambda_4^{-1})^{1/2}$ we obtain

$$\mu = -t^{9/2} \qquad \alpha = t^{3/2} + t^{1/2} + t^{-1/2} + t^{-3/2}.$$

The result coincides with those of Wadati *et al* [23]. Under the case E is given by equation (3.1) with

$$r_a = t^{-(1/2)a} \qquad a \in [-\tfrac{3}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}].$$

This example also checks our general results.

Another example is BGR associated with 7-dimensional representation of G_2 which possesses the typical structure to yield equation (2.10). In this case the four distinct eigenvalues are given by

$$\lambda_1 = -u^{-3} \qquad \lambda_2 = u \qquad \lambda_3 = -1 \qquad \lambda_4 = u^{-6}.$$

By taking $\mu = (-\lambda_1^2\lambda_3^3\lambda_4^{-1})^{1/2}$ we obtain

$$\mu = 1 \qquad \alpha = u^5 + u^4 + u + 1 + u^{-1} + u^{-4} + u^{-5}$$

and

$$r_a = \varepsilon_a u^{\tilde{a}}$$

where $a \in [-3, -2, -1, 0, 1, 2, 3]$ and $\varepsilon_a = 1$ $(a = -3, -1, 3)$, -1 $(a = -2, 0, 2)$. $\tilde{a}$ is defined by

$$\tilde{a} = \begin{cases} a - \frac{1}{2} & (a = 3, 1) \\ a & (a = 2, 0, -2) \\ a + \frac{1}{2} & (a = -1, -3). \end{cases}$$

This BGR can be Yang-Baxterized to satisfy equation (2.1) through equation (2.13), which have been shown in [25].

It is expected to find more examples obeying $\mathscr{A}$ and set up the connection with link polynomials.

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